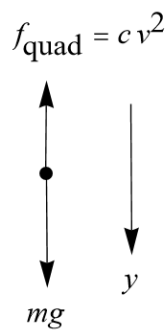
Problem 2.29

The terminal speed of a 70-kg skydiver in spread-eagle position is around 50 m/s (about 115 mi/h). Find his speed at times $t = 1, 5, 10, 20, 30$ seconds after he jumps from a stationary balloon. Compare with the corresponding speeds if there were no air resistance.

Solution

Draw a free-body diagram for a skydiver falling down in a medium with quadratic air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - cv^2 = m\frac{dv}{dt} \tag{1}$$

The terminal speed occurs when the velocity reaches equilibrium.

$$mg - cv_{\text{ter}}^2 = m(0)$$

Solve for v_{ter}, the terminal velocity.

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

To get v, solve equation (1) by separating variables.

$$c\left(\frac{mg}{c} - v^2\right) = m\frac{dv}{dt}$$

$$\frac{c}{m}\,dt = \frac{dv}{\frac{mg}{c} - v^2}$$

Integrate both sides definitely, assuming that at $t = 0$ the skydiver's velocity is zero (stationary balloon).

$$\int_0^t \frac{c}{m}\,dt' = \int_0^v \frac{dv'}{\frac{mg}{c} - v'^2} \tag{2}$$

Make the following substitution in the integral on the right side.

$$v' = \sqrt{\frac{mg}{c}} \sin\theta$$

$$dv' = \sqrt{\frac{mg}{c}} \cos\theta \, d\theta$$

Consequently, equation (2) becomes

$$\frac{c}{m}(t-0) = \int_{\sin^{-1}\left(\frac{0}{\sqrt{\frac{mg}{c}}}\right)}^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\sqrt{\frac{mg}{c}} \cos\theta \, d\theta}{\frac{mg}{c}(1-\sin^2\theta)}$$

$$\frac{c}{m} t = \frac{1}{\sqrt{\frac{mg}{c}}} \int_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\cos\theta \, d\theta}{\cos^2\theta}$$

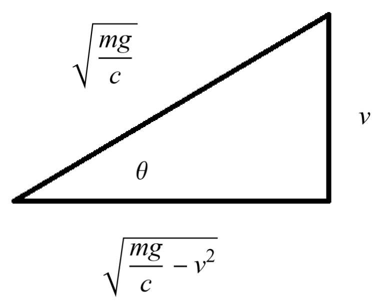
$$= \sqrt{\frac{c}{mg}} \int_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \sec\theta \, d\theta$$

$$= \sqrt{\frac{c}{mg}} \ln|\sec\theta + \tan\theta| \Big|_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)}$$

$$= \sqrt{\frac{c}{mg}} \ln\left[\frac{\sec\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) + \tan\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)}{\sec 0 + \tan 0}\right]$$

$$= \sqrt{\frac{c}{mg}} \ln\left[\sec\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) + \tan\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)\right].$$

Draw the implied right triangle.



As a result, the formula simplifies to

$$\begin{aligned}
\frac{c}{m}t &= \sqrt{\frac{c}{mg}}\ln\left(\frac{\sqrt{\frac{mg}{c}}}{\sqrt{\frac{mg}{c}-v^2}}+\frac{v}{\sqrt{\frac{mg}{c}-v^2}}\right) \\
&= \sqrt{\frac{c}{mg}}\ln\left(\frac{\sqrt{\frac{mg}{c}}+v}{\sqrt{\frac{mg}{c}-v^2}}\right) \\
&= \sqrt{\frac{c}{mg}}\ln\left[\frac{\sqrt{\frac{mg}{c}}+v}{\sqrt{\left(\sqrt{\frac{mg}{c}}+v\right)\left(\sqrt{\frac{mg}{c}}-v\right)}}\right] \\
&= \sqrt{\frac{c}{mg}}\ln\sqrt{\frac{\sqrt{\frac{mg}{c}}+v}{\sqrt{\frac{mg}{c}}-v}} \\
&= \sqrt{\frac{c}{mg}}\left[\frac{1}{2}\ln\left(\frac{1+\frac{v}{\sqrt{\frac{mg}{c}}}}{1-\frac{v}{\sqrt{\frac{mg}{c}}}}\right)\right] \\
&= \sqrt{\frac{c}{mg}}\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right).
\end{aligned}$$

Now solve for v.

$$\sqrt{\frac{cg}{m}}\,t = \tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)$$

$$\tanh\left(\sqrt{\frac{cg}{m}}\,t\right) = \frac{v}{\sqrt{\frac{mg}{c}}}$$

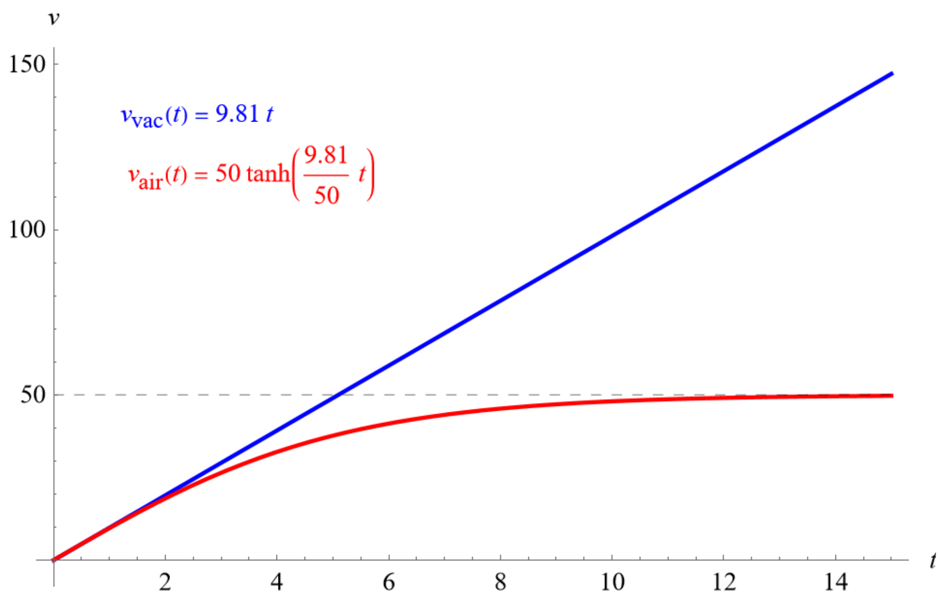
Therefore, the skydiver's velocity in a medium with quadratic air resistance is

$$v_{\text{air}}(t) = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right) = v_{\text{ter}}\tanh\left(\frac{g}{v_{\text{ter}}}t\right) = 50\tanh\left(\frac{9.81}{50}t\right),$$

whereas the velocity in a vacuum is just

$$mg = m\frac{dv}{dt} \quad\rightarrow\quad \frac{dv}{dt} = g \quad\rightarrow\quad v_{\text{vac}}(t) = gt = 9.81t.$$

Below is a graph (in SI units) of the two velocities versus time.



The velocities are calculated and compared at

$$t = 1: \qquad \left.\begin{aligned} v_{\text{vac}} &= 9.81\ \frac{\text{m}}{\text{s}} \\ v_{\text{air}} &\approx 9.69\ \frac{\text{m}}{\text{s}} \\ \text{Percent Difference} &\approx 1.28\% \end{aligned}\right\}$$

$$t = 5: \qquad \left.\begin{aligned} v_{\text{vac}} &\approx 49.1\ \frac{\text{m}}{\text{s}} \\ v_{\text{air}} &\approx 37.7\ \frac{\text{m}}{\text{s}} \\ \text{Percent Difference} &\approx 30.2\% \end{aligned}\right\}$$

$$t = 10: \qquad \left.\begin{aligned} v_{\text{vac}} &\approx 98.1\ \frac{\text{m}}{\text{s}} \\ v_{\text{air}} &\approx 48.1\ \frac{\text{m}}{\text{s}} \\ \text{Percent Difference} &\approx 104\% \end{aligned}\right\}$$

as well as

$$t = 20: \qquad \left.\begin{aligned} v_{\text{vac}} &\approx 196\ \frac{\text{m}}{\text{s}} \\ v_{\text{air}} &\approx 50.0\ \frac{\text{m}}{\text{s}} \\ \text{Percent Difference} &\approx 293\% \end{aligned}\right\}$$

$$t = 30: \qquad \left.\begin{aligned} v_{\text{vac}} &\approx 294\ \frac{\text{m}}{\text{s}} \\ v_{\text{air}} &\approx 50.0\ \frac{\text{m}}{\text{s}} \\ \text{Percent Difference} &\approx 489\% \end{aligned}\right\}.$$

The percent difference at time t is calculated by

$$\text{Percent Difference} = \frac{v_{\text{vac}}(t) - v_{\text{air}}(t)}{v_{\text{air}}(t)} \times 100\%.$$